

On relativization of the Sommerfeld-Gamow-Sakharov factor

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ABSTRACT: The Sommerfeld-Gamow-Sakharov factor is considered for the general case of arbitrary masses and energies. It is shown that the scalar triangular one-loop diagram gives the Coulomb singularity in radiative corrections at the threshold. The singular part of the correction is factorized at the complete Born cross section regardless of its partial wave decomposition. Different approaches to generalize the factor are discussed.

KEYWORDS: Coulomb singularity, final state interactions, rescattering

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1 Introduction

It is well known that the electromagnetic interaction between charged particles in the final state can considerably affect the observable reaction rate. For example, the cross section of electron-positron annihilation into muons becomes different from zero at the threshold due to the final state interactions. Another observable effect is the difference in energy behavior at the threshold of the annihilation channels with production of charged and neutral mesons, see paper [1] and references therein. In the case of strong interactions in the final state, e.g. in the processes with creation of a heavy quark pair, a similar nonperturbative enhancement factor appears [2]. It was shown [3] that interplay of both QED and QCD final state interactions can be also important. The effects of the Coulomb singularity in production of stop and gluino pairs close to their thresholds were discussed in Ref. [4].

If the relative velocity of the charged particles is small ($v \ll 1$)¹, then the effect of multiple photon exchange between them becomes significant. This fact has been discussed in the literature for a long time. It was shown already in the textbook by A. Sommerfeld [5] that the correction due to re-scattering of charged particles in the final state is proportional to the wave function at the origin squared, $|\Psi(0)|^2$, see also book [6]. So that the scattering (or annihilation) channel acquires some features of the corresponding bound state. G. Gamow has shown [7] that the same factor is relevant for the description of the Coulomb barrier in nuclear interactions. Using the non-relativistic Schrödinger equation, A. Sakharov derived this factor for the case of charged pair production [8] in the form

$$T = \frac{\eta}{1 - e^{-\eta}}, \quad \eta = \frac{2\pi\alpha}{v}, \quad (1.1)$$

where $\alpha \approx 1/137$ is the fine structure constant and v is the (non-relativistic) relative velocity of the particles in the created pair,

$$v = \left| \frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2} \right|. \quad (1.2)$$

Here $\vec{p}_{1,2}$ and $m_{1,2}$ are the momenta and masses of the particles.

The behavior of the factor is shown in Fig. 1

¹We use natural units $c = \hbar = 1$.

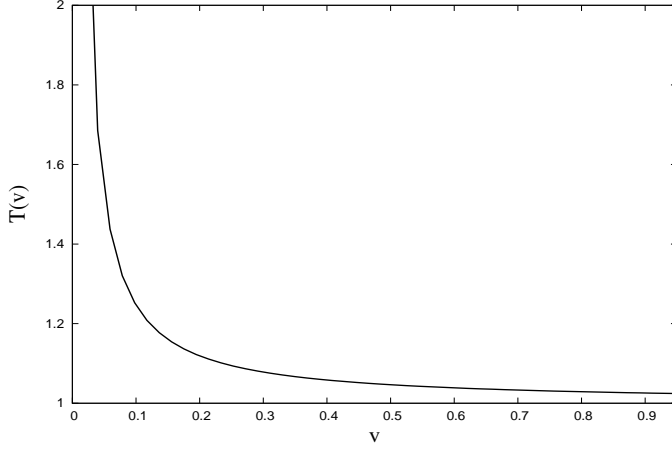


Figure 1. SGS factor in QED vs. the relative velocity.

The question about relativization of the Sommerfeld-Gamow-Sakharov (SGS) factor and about some other ways of its generalization, e.g. for non-equal masses and P -waves, is under discussion in the literature for a long time, see papers [9–16] and references therein. In ref. [17] the possibility to generalize the factor by inclusion of strong and weak interactions was studied.

2 SGS factor

Let us first discuss the general features of the SGS factor. It is useful to consider the limit of a small coupling constant²

$$\lim_{\eta \rightarrow 0} T = 1 + \frac{\eta}{2} + \frac{\eta^2}{12} + \mathcal{O}(\eta^3) = 1 + \frac{\pi\alpha}{v} + \frac{\pi^2\alpha^2}{6v^2} + \mathcal{O}((\alpha/v)^3). \quad (2.1)$$

In this way we get terms which can be related to the ones arising in a perturbative calculation.

Formula (1.1) can be easily adapted for the case of arbitrary charges Q_1 and Q_2 by taking $\eta = -Q_1Q_2 \cdot 2\pi\alpha/v$. The fact that for the repulsion case ($Q_1Q_2 > 0$) there is no Coulomb singularity provides an asymmetry in contributions of different quarks pairs taken from the final state hadrons. This allows to discuss in refs. [18, 19] the possibility the have a threshold enhancement factor even for the neutral baryon ($\Lambda\bar{\Lambda}$) production case.

Let us consider the case of the final state interaction³ of two charged particles produced close to the threshold, e.g. in electron-positron annihilation

$$e^-(k_1) + e^+(k_2) \rightarrow a^-(p_1) + a^+(p_2), \quad (2.2)$$

$$s = (k_1 + k_2)^2 = (p_1 + p_2)^2 \gtrsim (m_1 + m_2)^2, \quad (2.3)$$

²Obviously, this expansion can not be used if $v \leq 2\pi\alpha$.

³A very similar picture takes place for the case of initial state interactions.

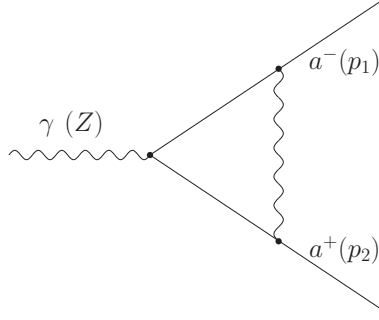


Figure 2. Feynman diagram for one-loop virtual correction in the final state.

where a^\pm can be scalar, spinor, or vector particles. The Born-level cross section σ^{Born} of this process depends on the type of integration and spin. But in any case in the center-of-mass system, it is proportional to the first power of factor $\beta_{1,2}$ which comes from the phase space volume and vanishes at the threshold $s \rightarrow (m_1 + m_2)^2$,

$$\beta_{1,2} = \frac{2p}{p_1^0 + p_2^0}, \quad p \equiv |\mathbf{p}_1| = |\mathbf{p}_2| = \frac{\sqrt{\Lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}},$$

$$p_1^0 + p_2^0 = 2\sqrt{s}, \quad \Lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (2.4)$$

For the case of equal masses this factor takes the usual form of the relativistic velocity $\beta = \sqrt{1 - m^2/(p^0)^2}$ of the final state particles.

In the one-loop approximation the QED radiative correction to the annihilation cross section gets contributions from virtual and real photon emission:

$$\sigma^{1\text{-loop}} = \sigma^{\text{Born}} \left(1 + \delta^{\text{Virt}} + \delta^{\text{Real}} \right). \quad (2.5)$$

The last term in the parentheses is proportional to β^2 , it is strongly suppressed at the threshold. Explicit expressions for different contributions with the exact dependence on the final state particle mass (for the equal mass case) can be found e.g. in ref. [20] for fermions and in ref. [21] for scalars. The final state one-loop virtual correction (in the on-mass-shell renormalization scheme) is described by the triangle diagram shown in Fig. 2. There are three types of integrals over the loop momentum: the scalar, the vector and the tensor ones:

$$\{I_S, I_V^\mu, I_T^{\mu\nu}\} = \int \frac{d^4k}{i\pi^2} \frac{\{1, k^\mu, k^{\mu\nu}\}}{((p_1 + k)^2 - m_1^2 + i\varepsilon)((p_2 - k)^2 - m_2^2 + i\varepsilon)(k^2 + i\varepsilon)}.$$

The tensor one contains an ultraviolet divergence, which has to be removed by the standard renormalization procedure. The vector integral I_V^μ is finite. The scalar integral I_S contains an infrared divergence, which cancels out in the sum with the contribution of the real final state radiation.

Direct calculations show that the contributions of the vector and tensor integrals are suppressed by at least the first power of the final state particle velocities. While the scalar

integral is proportional to $1/\sqrt{\Lambda(s, m_1^2, m_2^2)}$ and reveals at the threshold the well known Coulomb singularity. The coefficients before the integrals depend in general on the type of the particles, but the factor standing at the scalar integral is *universal*, it is the same for scalar, spinor and vector final state particles. The contributions of the one-loop scalar integral to the cross section can be presented in the form

$$\delta\sigma_S^{1\text{-loop}} = \sigma^{\text{Born}} \frac{\alpha}{\pi} Q_1 Q_2 (s - m_1^2 - m_2^2) I_s, \quad (2.6)$$

$$I_s \equiv C_0(m_1^2, m_2^2, s, m_1^2, m_1^2, m_2^2),$$

where the notation of the LoopTools package [22] for the Passarino-Veltman functions is used. The explicit form of this integral can be found for example in ref. [12]. The infrared divergence of this integral can be regularized by a fictitious photon mass m_γ or with the help of any other regularization scheme.

At the threshold, the integral takes the simple form

$$\lim_{s \rightarrow (m_1 + m_2)^2} I_s = \frac{1}{\sqrt{\Lambda(s, m_1^2, m_2^2)}} \left[-\pi^2 + \mathcal{O} \left(\sqrt{\frac{s - (m_1 + m_2)^2}{s}} \right) \right]. \quad (2.7)$$

Comparison of the order α term in Eq. (2.1) with the one obtained above gave us a hint to make the choice of SGS factor generalization. Namely, we see that the one-loop calculation is consistent with the substitution of the non-relativistic relative velocity by its relativistic version

$$v_{\text{rel}} = \frac{\sqrt{\Lambda(s, m_1^2, m_2^2)}}{s - m_1^2 - m_2^2} = \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{s - m_1^2 - m_2^2}. \quad (2.8)$$

We would like to underline that v_{rel} is exactly the relativistic sum of the velocities of our particles. This quantity is a relativistic invariant. For $s \gg m_{1,2}^2$ in the ultra-relativistic limit $v_{\text{rel}} \rightarrow 1$.

In the limiting case when one of the masses is heavy and the other is light, the relative velocity coincides with the one of the light particle (in the rest reference frame of the heavy particle). In this case the relativized SGS factor emerges from the relativistic one-particle Dirac (or Klein–Gordon–Fock) equation in a central field.

3 Discussion

The SGS factor is a part of radiative corrections which are used in the analysis of modern experimental data on various processes. To avoid a double counting we should *match* the factor with other higher order QED contributions. We suggest to perform the matching in the following way:

$$\begin{aligned} \sigma^{\text{Corr.}} &= \sigma^{\text{Born}} \left(T(v) - \frac{\pi\alpha}{v} - \frac{\pi^2\alpha^2}{3v^2} - \dots \right) \\ &+ \Delta\sigma^{1\text{-loop}} + \Delta\sigma^{2\text{-loop}} + \dots \end{aligned}$$

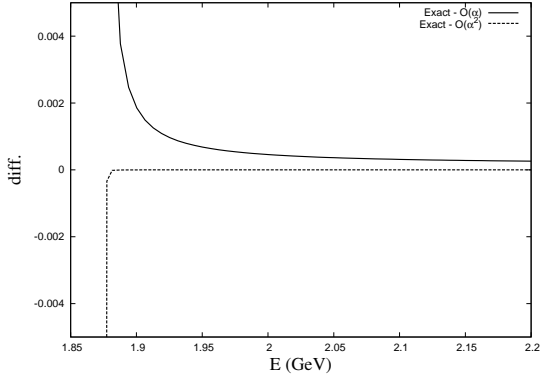


Figure 3. Difference between the resummed SGS factor and its perturbative expansion in different approximations.

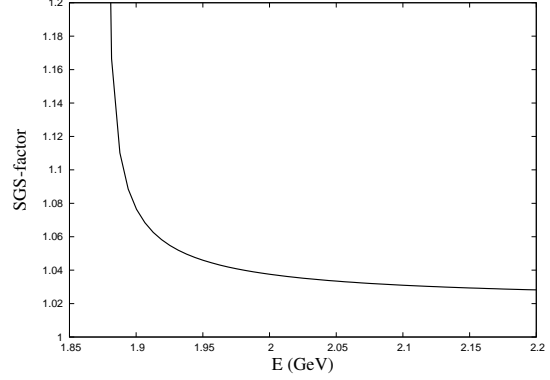


Figure 4. SGS factor vs. the center-of-mass energy for $e^+e^- \rightarrow p\bar{p}$.

Here $\Delta\sigma^{n-loop}$ is the n -th loop perturbative QED contribution to the observed (corrected) cross section $\sigma^{Corr.}$. The Born level cross section σ^{Born} can be taken in an *improved* approximation: it may include some higher order effects not related (to interactions in the final state), *e.g.* the vacuum polarization and the initial state radiation. So we subtract from the factor the first orders of its perturbative expansion, which are then restored by adding the explicit perturbative results (in the same orders).

Fig. 3 shows the difference between the complete SGS factor (1.1) and its perturbative expansion (2.1) for the process $e^+e^- \rightarrow p\bar{p}$ as a function of the center-of-mass energy. One can see that the difference is steeply rising up (or going down) in the region close to the threshold where the perturbative expansion breaks down. Nevertheless, soon above the threshold the difference becomes small especially for the case of the $\mathcal{O}(\alpha^2)$ approximation.

Let us compare our version of the generalized SGS factor with the other ones known in the literature. First of all, we note that the expression of the factor obtained here by extrapolation of the one-loop result coincides with the one derived in ref. [13] for the case of scalar particles with the help of a relativistic quasi-potential equation. Analyzing another relativistic quasipotential equation suggested by I. Todorov in ref. [23] (see eq. (4.1) there) also gives the same value of the wave function in the origin and thus the same version of the relativized SGS factor.

In ref. [9] resummation of ladder diagrams in the final state interactions of equal-mass fermions was performed. It was explicitly demonstrated that the form (1.1) of the SGS factor is reproduced. But instead of the non-relativistic relative velocity quantity 2β emerged (the same as in the *ad hoc* relativization procedure). To our mind, the reason for this is as follows. Keeping only the ladder diagram contribution without crossed photon lines corresponds to the pure Coulomb photon exchange, while in the relativistic case its contribution has the same order as the one due to transverse photons. It has been demonstrated in ref. [13] within a quasi-potential relativistic equation approach that keeping only the Coulomb interaction in the potential leads to $v = 2\beta$ while adding the transverse photon exchange restores the complete value of the relativistic relative velocity. Note also that the

ad hoc relativized version of the SGS factor ($v = 2\beta$) has a wrong ultra-relativistic limit $v \rightarrow 2$. Moreover, this version of the factor is not relativistic invariant.

Our version of the SGS factor is also supported in ref. [24], where the explicit analytical results for final state QED corrections to production of spinor particle with equal masses were considered. It was shown that the relativistic relative velocity

$$v = \frac{2\beta}{1 + \beta^2} \quad (3.1)$$

naturally appears in the case considered.

An original version of the relativized SGS factor for the case of arbitrary masses was derived in ref. [16] with the help of a relativistic two-body equation. This version of the factor satisfies the main condition: the non-relativistic expression is reproduced at the threshold. But the ultra-relativistic limit and the heavy-light mass ($m_2 \gg m_1$) one for v are missed.

Authors of ref. [25] claimed that the experimental data on the process $e^+e^- \rightarrow p\bar{p}$ favor application of the SGS factor in the threshold region without its denominator

$$T \Big|_{v \ll 1} \approx \mathcal{E} = \frac{\pi\alpha}{\beta}. \quad (3.2)$$

In this case the multiplier

$$\mathcal{R} = \frac{1}{1 - e^{-\pi\alpha/\beta}} \quad (3.3)$$

called there as the *resummation factor* is dropped. We would like to note, that the introduction of the resummation factor looks rather artificial in the view of the perturbative expansion (2.1), i.e. the enhancement factor (3.2) contains itself a certain nonperturbative (resummed) contribution. Moreover, neither the known one-loop QED corrections, nor the advocated above form of the relativized SGS factor were applied there.

In ref. [26] higher order effects due to the fine structure constant running were taken into account (see eq.(18) there) in the form of an additional factor. It is clear that this effect becomes numerically important only for ultra-relativistic relative velocities. The additional factor derived in [26] can be also applied for our version of the SGS factor. Another possibility is to take into account the running of α in the perturbative part of the matching formula (3.1).

In ref. [12] application of the SGS factor to production of unstable charged particles (a W^\pm pair) was considered. The authors also evaluated the one-loop triangular diagram. The factor $(s - m_1^2 - m_2^2)$ before the one-loop integral, see (2.6), was approximated there to be equal $s/2$. For this reason, their expression for the *relativized* relative velocity (standing in the SGS factor) has the correct non-relativistic limit but does not satisfy the ultra-relativistic and heavy-light mass limits.

Papers [27–29] also discuss production of W^+W^- boson pair near the threshold taking into account the width of W bosons and some higher order corrections. The relative velocity in the SGS factor was treated there in a non-relativistic manner: $v = v^+ + v^-$, where v^\pm was either relativistic or non-relativistic velocity of W^\pm in the c.m.s. We would like to underline that at the threshold such an approximation is very solid.

4 Conclusions

As discussed above there are several different approaches to generalize the Sommerfeld-Gamow-Sakharov factor. Most of them are equivalent from the practical point of view, since they differ by terms that vanish in the limit $v \rightarrow 0$. On the other hand, such terms are not universal (they depend on the choice of the process) and can not be re-summed in a unique way.

We demonstrated that there is a certain part of the final state QED correction which does not depend on spin of the interacting particles and appear in the order-by-order perturbative calculations exactly in the form of the non-relativistic factor expansion. The corresponding recipe of the SGS factor relativization consists just in the substitution of the non-relativistic relative velocity of the two particles by the relativistic one. This choice could have been suggested from the beginning, but actually we got it here by looking at one-loop perturbative radiative corrections. It is worth to underline that exactly the same relativized SGS factor was received in refs. [13, 23] with the help of relativistic quasi-potential equations.

The widely used version of the relativized SGS factor where the non-relativistic relative velocity is substituted by 2β (twice the velocity of a particle in the center-of mass frame) was criticized. In fact this version of the factor is not relativistic invariant and has a wrong ultra-relativistic limit.

As concerning phenomenological applications, we claim that any choice of the SGS factor which has the correct non-relativistic limit can be used. One should just take care on removing possible double counting if other (*e.g.* complete one-loop) radiative corrections are taken into account. The uncertainty due to the choice of the concrete SGS factor will lie then in uncontrolled terms of higher orders in α , which are not singular in the limit $v \rightarrow 0$.

For the case of non-equal masses we suggested to verify the heavy-light mass limit $m_1 \gg m_2$, where one can use for a cross check not only the solution of the non-relativistic Schrödinger equation but also the one of the relativistic Dirac equation. Our choice of the SGS factor satisfies this condition by construction.

It is worth noting that the SGS factor derived here is applicable for all partial waves which can appear in the process $\gamma^* \rightarrow ab$. In particular, it is factorized before the whole Born cross section of $e^+e^- \rightarrow p\bar{p}$ (before both the S and D wave contributions to it). For the case of pseudoscalar meson production, *e.g.* for $e^+e^- \rightarrow \pi^+\pi^-$, the same factor stands for p wave. Of course, the P and D waves Born-level contributions are proportional to higher powers of β , so there is no shift from 0 of the cross section at the threshold due to the Coulomb enhancement, but the radiative corrections themselves are large in that region.

The version of the SGS factor advocated here is implemented into Monte Carlo code MCGPJ [30] with matching to the complete first order corrections. Obviously, taking into account the resummation factor in the proper approximation would be important for the analysis of new precise data on production of particles near threshold are coming from experiments at VEPP2000 (Novosibirsk), BEPCII (Beijing), and other machines.

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